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Measures on Classes of Subspaces Affiliated with a von Neumann Algebra

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Abstract Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H and let S be a dense lineal in H that is affiliated with a von Neumann algebra $\mathcal{M}$. The "topological" definition of measures on the classes of orthoclosed and splitting subspaces of S affiliated with a von Neumann algebra $\mathcal{M}$ is given and results on the relationships of these measures to measures on orthoprojections of the algebra $\mathcal{M}$ are presented.

Keywords Inner product space · Von Neumann algebra · Orthoclosed and splitting subspaces · Measures on classes of subspaces

1 Introduction

Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space H and let S be a dense lineal in H that is *affiliated* with a von Neumann algebra $\mathcal{M}$ ($S\eta\mathcal{M}$), i.e., S is invariant with respect to an arbitrary operator in the commutant $\mathcal{M}'$ of the algebra $\mathcal{M}$. We consider the following classes of subspaces of S:

$L_{\mathcal{M}}(S)$, the class of all closed subspaces of S affiliated with a von Neumann algebra $\mathcal{M}$;
$F_{\mathcal{M}}(S)$, the class of all *orthoclosed* subspaces of S affiliated with $\mathcal{M}$, i.e.,

$$F_{\mathcal{M}}(S) = \{X \subseteq S \mid X = X^{\perp\perp}, X\eta\mathcal{M}\};$$

$E_{\mathcal{M}}(S)$, the class of all *splitting* subspaces of S affiliated with $\mathcal{M}$, i.e.,

$$E_{\mathcal{M}}(S) = \{X \subseteq S \mid X \oplus X^{\perp} = S, X\eta\mathcal{M}\}.$$

Note that the following chain of inclusions takes place:

$$E_{\mathcal{M}}(S) \subseteq F_{\mathcal{M}}(S) \subseteq L_{\mathcal{M}}(S). \tag{1}$$

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For $X \subseteq S$, we denote by the symbol $[X]$ the closure of X in H. By $\mathcal{M}^{pr}$ we denote the class of all orthoprojections of a von Neumann algebra $\mathcal{M}$.

Lemma 1 *Let X be a subspace in S such that $X\eta\mathcal{M}$, p is a orthoprojection to $[X]$. Then $p \in \mathcal{M}$.*

Lemma 2 *Let $X \in E_{\mathcal{M}}(S)$. Then $H = [X] \oplus [X^{\perp}]$.*

Proposition 3

(i) $L_{\mathcal{M}}(S) = \{\mathcal{R}(p) \cap S \mid p \in M^{pr}\}$;
(ii) $F_{\mathcal{M}}(S) = \{\mathcal{R}(p) \cap S \mid p \in M^{pr}, [\mathcal{R}(p) \cap S] = \mathcal{R}(p), [\mathcal{R}(1-p) \cap S] = \mathcal{R}(1-p)\}$;
(iii) $E_{\mathcal{M}}(S) = \{\mathcal{R}(p) \cap S \mid p \in M^{pr}, \mathcal{R}(p) \cap S = pS\}$.

Proof (i) Let $X \in L_{\mathcal{M}}(S)$. Then $[X]$ is a subspace in H. Let p be a orthoprojection to $[X]$. By Lemma 1, $p \in \mathcal{M}^{pr}$. Then $X \subseteq S$, $X \subseteq [X] = \mathcal{R}(p)$, i.e, $X \subseteq \mathcal{R}(p) \cap S$. Since X is closed in S, it follows that $\mathcal{R}(p) \cap S = X$. To prove the reverse inclusion, note that $\mathcal{R}(p) \cap S$ is closed in S and $(\mathcal{R}(p) \cap S)\eta\mathcal{M}$. In fact, since $p \in \mathcal{M}^{pr}$, it follows that $upf = puf \in \mathcal{R}(p)$ $(\forall u \in \mathcal{M}'^{un},\ f \in H)$, i.e. $u(\mathcal{R}(p)) \subseteq \mathcal{R}(p)$ $(\forall u \in \mathcal{M}'^{un})$. Then $L_{\mathcal{M}}(S) = \{\mathcal{R}(p) \cap S \mid p \in M^{pr}\}$.

(ii) Let $X \in F_{\mathcal{M}}(S)$. Then $[X]$ is a subspace in H. Let p be a orthoprojection to $[X]$. Then $X = \mathcal{R}(p) \cap S$, i.e., $[\mathcal{R}(p) \cap S] = \mathcal{R}(p)$. In addition, $X^{\perp} = \mathcal{R}(I-p) \cap S$. In fact,

$$X^{\perp} = \{f \in S \mid \langle f, g\rangle = 0\ \forall g \in X = \mathcal{R}(p) \cap S\} = \{f \in S \mid \langle f, g\rangle = 0\ \forall g \in \mathcal{R}(p)\}$$
$$= \{f \in S \mid f \in (\mathcal{R}(p))^{\perp}\}.$$

Let $[X^{\perp}] = \mathcal{R}(q)$ $(q \in \mathcal{M}^{pr})$. Then $\mathcal{R}(p) \cap S = X = X^{\perp\perp} = \mathcal{R}(I-q) \cap S$. Consequently, $q = I - p$ and $\mathcal{R}(I-p) = [\mathcal{R}(I-p) \cap S]$. To prove the reverse inclusion, consider $X = \mathcal{R}(p) \cap S$, where $p \in \mathcal{M}^{pr}$, $[\mathcal{R}(p) \cap S] = \mathcal{R}(p)$, $[\mathcal{R}(1-p) \cap S] = \mathcal{R}(1-p)$. Then

$$X^{\perp} = \{f \in S \mid \langle f, g\rangle = 0\ \forall g \in \mathcal{R}(p) \cap S\} = \{f \in S \mid \langle f, g\rangle = 0\ \forall g \in \mathcal{R}(p)\},$$

i.e. $X^{\perp} = \mathcal{R}(I-p) \cap S$.

$$X^{\perp\perp} = \mathcal{R}(I - (I-p)) \cap S = \mathcal{R}(p) \cap S,$$

i.e. $X = X^{\perp\perp}$.

In addition, $X\eta\mathcal{M}$.

(iii) Let $X \in E_{\mathcal{M}}(S)$.Then $[X]$ is a subspace in H. Let p be a orthoprojection to $[X]$. Since $X \in E_{\mathcal{M}}(S)$, we have by Lemma 2

$$f = f_1 + f_2 \quad \forall f \in S,$$

where $f_1 \in X \subseteq [X]$, $f_2 \in X^{\perp} \subseteq [X^{\perp}]$. Then $pf = f_1 \in \mathcal{R}(p) \cap S$, i.e. $pS \subseteq \mathcal{R}(p) \cap S$ and $pS \subseteq X$. In addition, $\mathcal{R}(p) \cap S \subseteq pS$. In fact, $\forall f \in \mathcal{R}(p) \cap S$ $f = pf \in pS$. Thus, $\mathcal{R}(p) \cap S = pS$. Consequently, $X = \mathcal{R}(p) \cap S = pS$. To prove the reverse inclusion, consider $X = \mathcal{R}(p) \cap S = pS$, where $p \in \mathcal{M}^{pr}$. Then $X^{\perp} = \mathcal{R}(I-p) \cap S$. Thus, it suffices to prove that

$$S \subseteq (\mathcal{R}(p) \cap S) \oplus (\mathcal{R}(I-p) \cap S).$$

Let $f \in S$. Then $f = pf + (I - p)f$, where $pf \in pS = \mathcal{R}(p) \cap S$, $(I - p)f \in \mathcal{R}(I - p)f \cap S$ (S is lineal). Since $pf \perp (I - p)f$, we have $f \in (\mathcal{R}(p) \cap S) \oplus (\mathcal{R}(I - p) \cap S)$. Then $S = X \oplus X^{\perp}$, where $X = \mathcal{R}(p) \cap S$ and $(\mathcal{R}(p) \cap S)\eta\mathcal{M}$. □

Example 4 Let $\mathcal{M}$ be a commutative von Neumann algebra. Let us show that in this case we have $E_{\mathcal{M}}(S) = F_{\mathcal{M}}(S) = L_{\mathcal{M}}(S)$. Let $X \in L_{\mathcal{M}}(S)$ and $p \in M^{pr}$ be such that $X = \mathcal{R}(p) \cap S$. We clearly have $X \subseteq pS$. Conversely, since $\mathcal{M} \subseteq \mathcal{M}'$ and $S\eta M$, it follows that $pS\eta M$, $pS \subseteq S$, and thus $pS \subseteq \mathcal{R}(p) \cap S = X$. It remains to apply Proposition 3(iii).

2 The Representation Space of a von Neumann Algebra Associated with a Weight

Let φ be a faithful normal semifinite weight on a semifinite von Neumann algebra $\mathcal{M}$. Denote by $\mathfrak{H}$ the Hilbert space being the completion $n_\varphi = \{x \in \mathcal{M} \mid \varphi(x^*x) < +\infty\}$ with respect to the scalar product $\langle \tilde{x}, \tilde{y} \rangle \equiv \varphi(y^*x)$ $(x, y \in n_\varphi)$. (Here $\tilde{x}$ is the image of x under the identity embedding n_φ into $\mathfrak{H}$.) Then the mapping $\pi_\varphi : \mathcal{M} \to \mathcal{B}(\mathfrak{H})$, defined by $\pi_\varphi(x)\tilde{y} \equiv \widetilde{xy}$ $(x \in \mathcal{M}, y \in n_\varphi)$, determines an $*$-representation of the algebra $\mathcal{M}$ into the algebra $\mathcal{B}(\mathfrak{H})$.

Let $\mathfrak{M} = \pi_\varphi(\mathcal{M})$ be the image of $\mathcal{M}$ under the canonical representation associated with φ. Then π_φ is an isomorphism of the von Neumann algebra $\mathcal{M}$ onto the von Neumann algebra $\mathfrak{M}$. Let in addition φ_π be the weight obtained by transferring of the weight φ to the algebra $\pi_\varphi(\mathcal{M})$: $\varphi_\pi(\pi_\varphi(x)) \equiv \varphi(x)$ $(x \in \mathcal{M}^+)$.

By D_{φ_π} we denote the lineal of the weight φ_π. In accordance with the general theory of Hilbert algebras, there is associated with a von Neumann algebra $\mathcal{M}$ the left Hilbert algebra $\mathfrak{A} = \{\tilde{x} \mid x \in n_\varphi \cap n_\varphi^*\}$. Consider

$$\mathfrak{A}'_0 = \{f \in \mathfrak{H} \mid \text{the mapping } \tilde{x} \to \pi_\varphi(x)f \ (x \in n_\varphi) \text{ is continuous}\},$$

the set of bounded from the right elements with respect to the left Hilbert algebra $\mathfrak{A}$. It is known that $\mathfrak{A}'_0 = \mathcal{D}_{\varphi_\pi}$. The lineal $\mathcal{D}_{\varphi_\pi}$ is affiliated with $\mathfrak{M}$ and is dense in $\mathfrak{H}$.

Thus, we consider the following situation: a von Neumann algebra $\mathfrak{M}$ acts in a Hilbert space $\mathfrak{H}$. Let $S \equiv \mathcal{D}_{\varphi_\pi}$ be a dense in $\mathfrak{H}$ lineal affiliated with $\mathfrak{M}$.

Theorem 5 *If, in the framework of the described construction, φ is a trace, then*

$$E_{\mathfrak{M}}(S) = F_{\mathfrak{M}}(S) = L_{\mathfrak{M}}(S).$$

Proof Note first that any orthoprojector from $\mathfrak{M}$ can be represented in the form $\pi_\varphi(p)$, where $p \in \mathcal{M}^{pr}$. Since φ is a trace, the set n_φ is a two-sided ideal. Let us show that $\pi_\varphi(p)S \subseteq S$ for any $p \in \mathcal{M}^{pr}$. Let $f \in S$, i. e., the mapping $\tilde{x} \to \pi_\varphi(x)f$ is continuous $(x \in n_\varphi)$. In particular, the mapping $\tilde{xp} \to \pi_\varphi(xp)f$ is continuous (because n_φ is a two-sided ideal). Since

$$\|\widetilde{xp}\|^2 = \langle \widetilde{xp}, \widetilde{xp} \rangle = \varphi(px^*xp) = \varphi(xpx^*) \le \varphi(xx^*) = \|\tilde{x}\|^2,$$

it follows that the mapping $\tilde{x} \to \tilde{xp}$ is continuous. Since the mapping $\tilde{xp} \to \pi_\varphi(xp)f$ is continuous, it follows that the mapping $\tilde{x} \to \pi_\varphi(x)\pi_\varphi(p)f (= \pi_\varphi(xp)f)$ is continuous. Thus, $\pi_\varphi(p)f \in S$ for $f \in S$. Hence it follows that any subspace S affiliated with $\mathfrak{M}$ is splitting, i.e. $E_{\mathfrak{M}}(S) = F_{\mathfrak{M}}(S) = L_{\mathfrak{M}}(S)$. □

Remark If we consider the algebra of all bounded operators $\mathcal{B}(H)$ as a von Neumann algebra and the standard trace τ_0 on $\mathcal{B}(H)$), then

$$n_{\tau_0} = \{x \in \mathcal{B}(H) \mid \tau_0(x^*x) < +\infty\}$$

coincides with the set of Hilbert-Schmidt operators acting in H. As is known, the set of Hilbert-Schmidt operators endowed with the scalar product $\langle \tilde{x}, \tilde{y}\rangle \equiv \tau_0(y^*x)$ $(x, y \in n_{\tau_0})$ is a Hilbert space, i.e., in the framework of the construction under consideration, the unitary space S is complete. Consequently, $E_{\mathfrak{M}}(S) = F_{\mathfrak{M}}(S) = L_{\mathfrak{M}}(S)$.

Consider in detail the construction described above in the case of the algebra $\mathcal{B}(H)$ of all bounded linear operators in a separable Hilbert space H. Let φ be a faithful normal semifinite weight on the von Neumann algebra $\mathcal{B}(H)$ of all bounded linear operators in a separable Hilbert space H. Let $k \geq 0$ be a self-adjoint operator which is defined [4] by

$$\varphi(x) = \tau_0(kx).$$

(Here τ_0 is the standard trace on $\mathcal{B}(H)$.) We let

$$n_\varphi = \{x \in \mathcal{B}(H) \mid \varphi(x^*x) < +\infty\} = \{x \in \mathcal{B}(H) \mid \tau_0(kx^*x) < +\infty\}.$$

Proposition 6 *Let φ be a faithful normal semifinite weight on the algebra $\mathcal{B}(H)$ and k the nonnegative self-adjoint operator defined by* (1). *An operator x belongs to n_φ if and only if $k^{1/2}x^* \in \mathfrak{C}_2(H)$. (Here $\mathfrak{C}_2(H)$ is the set of Hilbert-Schmidt operators acting in the Hilbert space H).*

Proof Necessity. Let $x \in n_\varphi$. Show first that $\mathcal{D}(k^{1/2}x^*) = H$. Let $f \in H$ be arbitrary. We have $\langle k_\varepsilon x^* f, x^* f\rangle = \langle x k_\varepsilon x^* f, f\rangle \leq \|f\|^2 \cdot \tau_0(xk_\varepsilon x^*)$. Then $\sup_\varepsilon \langle k_\varepsilon x^* f, x^* f\rangle \leq \|f\|^2 \sup_\varepsilon \tau_0(k_\varepsilon x^* x) = \|f\|^2 \varphi(x^*x) < +\infty$ and $x^* f \in \mathcal{D}(k^{1/2})$. Therefore, $f \in \mathcal{D}(k^{1/2}x^*)$. Thus, $\mathcal{D}(k^{1/2}x^*) = H$. Since $k^{1/2}$ is closed, it follows that $k^{1/2}x^*$ is also closed. But a closed operator defined on the whole Hilbert space is bounded. Let (e_i) be an orthonormal basis in H. Then, for any $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} \langle k^{1/2}x^*e_i, k^{1/2}xe_i\rangle = \lim_{\varepsilon\downarrow 0} \sum_{i=1}^{n} \|k_\varepsilon^{1/2}x^*e_i\|^2 \leq \lim_{\varepsilon\downarrow 0} \sum_{i=1}^{+\infty} \|k_\varepsilon^{1/2}x^*e_i\|^2$$
$$= \lim_{\varepsilon\downarrow 0} \tau_0(xk_\varepsilon x^*) = \varphi(x^*x) < +\infty.$$

Since n an arbitrary number, it follows that $\sum_{i=1}^{+\infty} \|k^{1/2}x^*e_i\|^2 < +\infty$. Therefore, $k^{1/2}x^*$ is a Hilbert-Schmidt operator.

Sufficiency. Let $k^{1/2}x^*$ be a Hilbert-Schmidt operator. Since

$$\tau_0(kx^*x) = \lim_{\varepsilon\downarrow 0} \tau_0(k_\varepsilon x^*x) = \lim_{\varepsilon\downarrow 0} \tau_0(xk_\varepsilon x^*) = \lim_{\varepsilon\downarrow 0} \sum_{i=1}^{+\infty} \|k_\varepsilon^{1/2}x^*e_i\|^2 = \sum_{i=1}^{+\infty} \|k^{1/2}x^*e_i\|^2 < +\infty,$$

we have $\varphi(x^*x) = \tau_0(kx^*x) < +\infty$, i.e. $x \in n_\varphi$. □

Theorem 7 *The mapping $x \to \overline{xk^{1/2}}$ $(x \in n_\varphi)$ defines an isometric isomorphism between the Hilbert space $\mathfrak{H}$ and the Hilbert space $\mathfrak{C}_2$ of Hilbert-Schmidt operators in the initial space H.*

Proof Since $\overline{xk^{1/2}} = (k^{1/2}x^*)^*$, from Proposition 6 it follows that $\overline{xk^{1/2}} \in \mathfrak{C}_2$. By the definition of the scalar products in the spaces $\mathfrak{C}_2$ and $\overline{\mathfrak{H}}$ and by virtue of the completeness of these spaces, it suffices to show that the set $\mathfrak{N} = \{\overline{xk^{1/2}} \mid x \in n_\varphi\}$ is dense in $\mathfrak{C}_2$. Since any Hilbert-Schmidt operator can be represented in the form $z = \sum_j \lambda_j \langle \cdot, \xi_j\rangle \xi_j$, where (ξ_j) is an appropriate orthonormal system in H, and $\sum_j \lambda_j^2 < +\infty$, it follows that this operator can be approximated in the Hilbert-Schmidt norm by operators of the form $z_n = \sum_{j=1}^n \lambda_j \langle \cdot, \xi_j\rangle \xi_j$. Thus, it suffices to prove that an operator of the form $\langle \cdot, \xi\rangle \xi$ ($\xi \in H$, $\|\xi\| = 1$) can be approximated in the norm of the space $\mathfrak{C}_2$ by operators from $\mathfrak{N}$.

Since $\mathcal{R}(k^{1/2})$ is dense in H, there exists a sequence $(\eta_n) \subset D(k^{1/2})$ such that $k^{1/2}\eta_n \to \xi$ ($n \to +\infty$). We let $x_n^* = \langle \cdot, \xi\rangle \eta_n$. Then

$$\begin{aligned}\|\overline{x_n k^{1/2}} - \langle \cdot, \xi\rangle \xi\|_2^2 &= \|k^{1/2}x_n^* - \langle \cdot, \xi\rangle \xi\|_2^2 = \|\langle \cdot, \xi\rangle (k^{1/2}\eta_n - \xi)\|_2^2 \\ &= \|k^{1/2}\eta_n - \xi\|^2 \to 0.\end{aligned}$$ □

Proposition 8 *Any lineal X in S affiliated with $\mathfrak{M}$ is of the form*

$$X = \{k^{1/2}x^* \mid x \in i_\varphi\},$$

where i_φ is a left ideal in $\mathcal{B}(H)$, $J^* = \{x^* \in \mathcal{B}(H) \mid k^{1/2}x^* \in X\}$ is a right ideal in $\mathcal{B}(H)$.

For a lineal $X \subset S$ affiliated with $\mathfrak{M}$, we let $\Delta(X) = \bigcup_{x^* \in J^*} \mathcal{R}(x^*)$. Obviously, any operator z from X can be represented in the form

$$z = \sum_j \langle \cdot, \xi_j\rangle k^{1/2}\eta_j, \quad \eta_j \in \Delta(X), \tag{2}$$

where (ξ_j) is an appropriate orthonormal system in H, $(k^{1/2}\eta_j)$ is an orthogonal system in H, and $\sum_j \|k^{1/2}\eta_j\|^2 < +\infty$.

We continue to study in detail the construction of the representation space of the algebra $\mathcal{B}(H)$ associated with a faithful normal semifinite weight assuming that the operator k is bounded.

If $X \in L_{\mathfrak{M}}(S)$, then $\Delta(X) = [\Delta(X)]$, where $[\Delta(X)]$ is the closure of $\Delta(X)$ in H. Furthermore, $z \in X$ if and only if representation (2) takes place.

Theorem 9 *If $X \in L_{\mathfrak{M}}(S)$, then there exists $p \in \mathcal{B}(H)^{pr}$ such that*

$$X = S_p \equiv \{k^{1/2}px^* \mid x^* = px^* \in n_\varphi^*\}.$$

Proof Let $p \equiv p_{\Delta(X)}$. Obviously, $X \subseteq S_p$.

To prove the reverse inclusion, consider

$$z = k^{1/2}x^* = k^{1/2}px^* \in S_p.$$

This operator admits representation (2) since $k^{1/2}x^*$ is a Hilbert-Schmidt operator. Then

$$\left\| z - \sum_{j=1}^{s} \langle \cdot, \xi\rangle k^{1/2}\eta_j \right\|_2^2 = \sum_{j=s+1}^{\infty} \|k^{1/2}\eta_j\|^2 \to 0 \quad (s \to +\infty)$$

(by virtue of the convergence of the series). Consequently, z can be approximated by finite-dimensional operators from X. But X is closed. Therefore, $z = k^{1/2}x^* \in X$. □

Theorem 10 *For $S_p \in L_{\mathfrak{M}}(S)$ the following conditions are equivalent*:

(i) $S_p \in E_{\mathfrak{M}}(S)$,
(ii) *there exists an orthoprojector $q \in \mathcal{B}(H)^{pr}$ with the properties*:

$$pkq = 0, \quad \mathcal{R}(p) + \mathcal{R}(q) = H.$$

Proof (i) $\Rightarrow$ (ii) Let $\eta \in H$ be arbitrary. Then $\langle \cdot, \xi\rangle k^{1/2}\eta \in S(\xi \in H, \|\xi\| = 1)$. Since $S = S_p \oplus S_q$ $(S_q = (S_p)^{\perp}, p, q \in \mathcal{B}(H)^{pr})$, it follows that $\langle \cdot, \xi\rangle k^{1/2}\eta = k^{1/2}px_1^* + k^{1/2}qx_2^*$ $(x_1^*, x_2^* \in n_\varphi^*)$. Then $k^{1/2}\eta = k^{1/2}px_1^*\xi + k^{1/2}qx_2^*\xi$, i.e. $k^{1/2}(\eta - px_1^*\xi - qx_2^*\xi) = 0$. Consequently, $\eta = px_1^*\xi + qx_2^*\xi \in \mathcal{R}(p) + \mathcal{R}(q)$. We have $H \subseteq \mathcal{R}(p) + \mathcal{R}(q) \subseteq H$.

(ii) $\Rightarrow$ (i). Let $S_p \in L_{\mathfrak{M}}(S)$, $pkq = 0$, $\mathcal{R}(p) + \mathcal{R}(q) = H$. We need to show that $S_p \in E_{\mathfrak{M}}(S)$. We let $(S_p)^{\perp} = S_q \in L_{\mathfrak{M}}(S)$. Let $\langle \cdot, \xi\rangle k^{1/2}\eta \in S$ be an arbitrary one-dimensional operator. Since $\eta = pf + qh$, $(f, h \in H)$, it follows that $\langle \cdot, \xi\rangle k^{1/2}\eta = \langle \cdot, \xi\rangle k^{1/2}pf + \langle \cdot, \xi\rangle k^{1/2}qh$. Since $pf \in \Delta(X)$, $qh \in \Delta(Y)$ (here $X = S_p$, $Y = S_q$), it follows that $\langle \cdot, \xi\rangle k^{1/2}pf \in S_p$, $\langle \cdot, \xi\rangle k^{1/2}qh \in S_q$,i.e. $S^{(1)} \subseteq S_p \oplus S_q$. Similarly, for $\sum_{i=1}^n \langle \cdot, \xi_i\rangle k^{1/2}\eta_i \in S^{(fin)}$ we have

$$\sum_{i=1}^n \langle \cdot, \xi_i\rangle k^{1/2}\eta_i = \sum_{i=1}^n \langle \cdot, \xi_i\rangle k^{1/2}pf_i + \sum_{i=1}^n \langle \cdot, \xi_i\rangle k^{1/2}qh_i \in S_p + S_q.$$

Then $S^{(fin)} \subseteq S_p \oplus S_q$.

Let now $z = \sum_i \langle \cdot, \xi_i\rangle k^{1/2}\eta_i \in S$ be arbitrary. We have i $\eta_i = pf_i + qh_i$ for any i. In addition, $\|k^{1/2}\eta_i\|^2 = \|k^{1/2}pf_i\|^2 + \|k^{1/2}qh_i\|^2$. Since $\sum_i \|k^{1/2}\eta_i\|^2 < +\infty$, it follows that $\sum_i \|k^{1/2}pf_i\|^2 < +\infty$, $\sum_i \|k^{1/2}qh_i\|^2 < +\infty$. Let $z_p = \sum_i \langle \cdot, \xi_i\rangle k^{1/2}pf_i$, $z_q = \sum_i \langle \cdot, \xi_i\rangle k^{1/2}qh_i$. Then

$$\begin{aligned}\|z_p\|_2^2 &= \sum_j \left\langle \sum_i \langle \xi_j, \xi_i\rangle k^{1/2}pf_i, \sum_k \langle \xi_j, \xi_k\rangle k^{1/2}pf_k \right\rangle \\ &= \sum_j \langle k^{1/2}pf_j, k^{1/2}pf_j\rangle = \sum_{i\in I} \|k^{1/2}pf_i\|^2 < +\infty\end{aligned}$$

and $z_p \in n_\varphi$. Similarly, $z_q \in n_\varphi$. □

3 Measures on Classes of Subspaces

Since there exists a certain correspondence between the subspaces of the unitary space affiliated with a von Neumann algebra and the orthogonal projectors from this algebra, the problem arises of introducing measures on the classes of subspaces under consideration and of studying relations between the measures on the classes of subspaces and the measures on the orthoprojections of the initial von Neumann algebra.

In what follows by $\mathcal{K}_{\mathcal{M}}(S)$ we denote one of the following classes $\{E_{\mathcal{M}}(S), F_{\mathcal{M}}(S), L_{\mathcal{M}}(S)\}$ and by $\bigoplus_{i\in I} X_i$ the closure of the set $\mathrm{lin}_{\mathbb{C}}\{X_i \mid i \in I\}$ in S for pairwise orthogonal subspaces from $L_{\mathcal{M}}(S)$.

Definition 11 A mapping $\mu : \mathcal{K}_{\mathcal{M}}(S) \to \mathbb{R}^+$ is called a *measure* (a *finitely additive measure*, and a *completely additive measure*, respectively) if, for any countable (finite, and arbitrary, respectively) family of mutually orthogonal subspaces $X_i \in \mathcal{K}_{\mathcal{M}}(S)$ $(i \in I)$, the condition $\bigoplus_{i\in I} X_i \in \mathcal{K}_{\mathcal{M}}(S)$ yields $\mu(\bigoplus_{i\in I} X_i) = \sum_{i\in I} \mu(X_i)$.

Proposition 12 *Let $m : \mathcal{M}^{pr} \to \mathbb{R}^+$ be a measure (a finitely additive measure, a completely additive measure, respectively). Then the relation*

$$\mu(X) \equiv m(p_{[X]}) \quad (X \in \mathcal{K}_{\mathcal{M}}(S))$$

defines a measure (a finitely additive measure, a completely additive measure, respectively) on the class $\mathcal{K}_{\mathcal{M}}(S)$.

Proof By the chain of inclusions (1), it suffices to prove the assertion for the class $L_{\mathcal{M}}(S)$. Let $\{X_i\} \subseteq L_{\mathcal{M}}(S)$ be a family of mutually orthogonal subspaces. Then $p_i \equiv p_{[X_i]}$ $(i \in I)$ are pairwise orthogonal orthoprojections in the algebra $\mathcal{M}$. Moreover, $p \equiv \sum_{i \in I} p_i = p_{[X]}$, where $X = \bigoplus_{i \in I} X_i$. Therefore,

$$\mu(X) = m(p_{[X]}) = \sum_{i \in I} m(p_i) = \sum_{i \in I} \mu(X_i),$$

where the second relation in the above chain holds for any countable (finite, and arbitrary, respectively) sum if m is a measure (a finitely additive measure, and a completely additive measure, respectively).

The inverse problem is in the study of conditions under which a measure (a finitely additive measure, and a completely additive measure, respectively) on the class $\mathcal{K}_{\mathcal{M}}(S)$ admits a "lifting" to a measure (a finitely additive measure, and a completely additive measure, respectively) on the orthoprojections of the algebra $\mathcal{M}$ is seemingly not very simple. Let us present two results in this direction. □

Proposition 13 *If $E_{\mathcal{M}}(S) = L_{\mathcal{M}}(S)$ and if μ is a measure (a finitely additive measure, and a completely additive measure, respectively) on $E_{\mathcal{M}}(S)$, then the relation*

$$m(p) = \mu(pS) \quad (p \in \mathcal{M}^{pr})$$

defines a measure (a finitely additive measure, and a completely additive measure, respectively) on $\mathcal{M}^{pr}$.

Proof By Proposition 3, the mapping $\mu : \mathcal{M}^{pr} \to \mathbb{R}^+$ is well defined on $E_{\mathcal{M}}(S)$. Let $p = \sum_{i \in I} p_i$, where $p_i p_j = 0$ $(i \neq j)$ and the set I is countable (finite, and arbitrary, respectively). Since

$$\left(\sum_{i \in I} p_i\right) S = \left(\bigoplus_{i \in I} p_i S\right),$$

it follows that

$$m(p) = \mu\left(\left(\sum_{i \in I} p_i\right) S\right) = \mu\left(\bigoplus_{i \in I} p_i S\right) = \sum_{i \in I} \mu(p_i S) = \sum_{i \in I} m(p_i).$$

Let S be a unitary space and let $E(S)$, $F(S)$, and $L(S)$ be the classes of splitting, orthoclosed, and closed subspaces of S, respectively. These classes are naturally identified with the classes $E_{\mathcal{B}(H)}(S)$, $F_{\mathcal{B}(H)}(S)$ and $L_{\mathcal{B}(H)}(S)$, where H is the Hilbert space that is the completion of S. □

Theorem 14 *Let S be an infinite-dimensional separable unitary space and let $\mu : \mathcal{K}_{\mathcal{B}(H)} \to \mathbb{R}^+$ be a measure. Then there exists a measure $m : \mathcal{B}(H)^{pr} \to \mathbb{R}^+$ that is uniquely defined by the condition*

$$m(\langle \cdot, f \rangle) = \mu(\mathrm{lin}_{\mathbb{C}}\{f\}) \quad f \in S, \ \|f\| = 1.$$

Proof Since all one-dimensional subspaces of S are splitting (and hence belong to $\mathcal{K}_{\mathcal{B}(H)}$), it follows that we can define a function on the unit sphere of the space S by the formula

$$\rho(f) \equiv \mu\big(\mathrm{lin}_{\mathbb{C}}\{f\}\big) \quad f \in S, \ \|f\| = 1.$$

This is a function of frame type in the Hilbert space H (the completion of S) [1], and thus is a restriction to the unit sphere of the space S of a positive quadratic form $t(f, f)$ ($f \in S$) that has a finite matrix trace [1]. By Theorem 4 in [1], this quadratic form is defined by a trace-class operator $T \geq 0$,

$$t(f, f) = \langle Tf, f \rangle \quad (f, g \in S).$$

The mapping $m : \mathcal{B}(H)^{pr} \to \mathbb{R}^+$ given by the relation

$$m(p) = \mathrm{tr}(Tp) \quad (p \in \mathcal{B}(H)^{pr})$$

is the desired measure. □

Remark It follows from the results of the note that Theorem 14 can be extended to complex-valued σ-additive charges.

We consider the representation space of the algebra $\mathcal{B}(H)$ associated with a faithful normal semifinite weight φ having bounded Radon-Nikodym derivative k, $\mathfrak{M} = \pi_\varphi(\mathcal{B}(H))$ is the image of $\mathcal{B}(H)$ under the indicated representation.

Theorem 15 *A measure given $\mathcal{K}_{\mathfrak{M}}(S)$ admits an "extension" to a measure on the orthoprojections of the algebra $\mathfrak{M}$.*

Proof Let $\mathcal{L} = \mathcal{R}(k^{1/2})$ be a dense in H lineal and $\mathcal{L}_1$ the unit sphere of the unitary space $\mathcal{L}$. If $k^{1/2}e \in \mathcal{L}_1$, then by p we denote the orthoprojection to $\mathrm{lin}_{\mathbb{C}}\{e\}$, i.e. $p = \frac{1}{\|e\|^2}\langle \cdot, e \rangle e$.

Let now $k^{1/2}e_1, k^{1/2}e_2 \in \mathcal{L}_1$, $\langle k^{1/2}e_1, k^{1/2}e_2 \rangle = 0$, and let p_1, p_2 be the orthoprojections corresponding to the vectors e_1, e_2. Let us show that $p_2 k p_1 = 0$. In fact, for any $f \in H$ we have

$$\begin{aligned} p_2 k p_1 f &= \frac{1}{\|e\|^2} p_2 k(\langle f, e_1 \rangle e_1) = \frac{\langle f, e_1 \rangle}{\|e\|^2} \langle k e_1, e_2 \rangle e_2 \\ &= \frac{\langle f, e_1 \rangle}{\|e\|^2} \langle k^{1/2} e_1, k^{1/2} e_2 \rangle e_2 = 0. \end{aligned}$$

In this case, the orthogonality condition for subspaces of the space S is fulfilled, i.e., $S_{p_1} \perp S_{p_2}$; in addition, $S_{p_1}, S_{p_2} \in E_{\mathfrak{M}}(S)$.

We define a function on $\mathcal{L}_1$ as follows: $\rho(k^{1/2}e) = \mu(S_p)$, $(k^{1/2}e \in \mathcal{L}_1)$. The definition is unambiguous because the equality $k^{1/2}e = k^{1/2}e^*$ implies that $k^{1/2}(e - e^*) = 0$, i. e., $e - e^* = 0$ since $Ker(k^{1/2}) = \{\theta\}$.

Let now $(k^{1/2}e_i)_{i\in I}$ be an arbitrary orthonormal system from $\mathcal{L}$. Then, for $p_i = \frac{1}{\|e_i\|^2}\langle\cdot, e_i\rangle e_i$ we have $S_{p_i}\perp S_{p_j}$ $(i \neq j)$; $S_{p_i} \in E_{\mathfrak{M}}(S)$. In this case,

$$\sum_{i\in I}\rho(k^{1/2}e_i) = \sum_{i\in I}\mu(S_{p_i}) = \mu(\oplus S_{p_i}) < +\infty.$$

In addition, if K is a finite-dimensional subspace in H, then the restriction ρ_K of ρ to $K\cap\mathcal{L}_1$ is a frame function in K. Thus, ρ is a frame type function in H [1]. Then ρ is the restriction to $\mathcal{L}_1$ of a positive quadratic form $t(f, f)$, $f \in \mathcal{L}$ with finite matrix trace. This implies the existence of a positive operator T, such that $t(f, f) = \langle Tf, f\rangle$, $f \in \mathcal{L}$.

The mapping $\overline{m} : \mathcal{B}(H)^{pr} \to \mathbb{R}^+$, given by $\overline{m}(p) = \mathrm{Tr}(Tp)$ $(p \in \mathcal{B}(H)^{pr})$, defines a measure on $\mathcal{B}(H)^{pr}$, and the equality $m(\pi_\varphi(p)) = \overline{m}(p)$ $(p \in \mathcal{B}(H)^{pr})$ defines a measure on $\mathfrak{M}^{pr}$. □

Remark The study of measures on the classes of closed subspaces of a unitary space was initiated by Hamhalter and Ptak in [3] who used the "orthomodular" definition of a measure on the class $F(S)$. It was assumed that the mapping $\mu : F(S) \to \mathbb{R}^+$ satisfies the condition

$$\mu\left(\bigvee_i X_i\right) = \sum_i \mu(X_i)$$

for any sequence of pairwise orthogonal subspaces $X_j \in F(S)$. It was established that the existence of a nonzero measure on the class $F(S)$ is equivalent to the completeness of the space S. A similar result for the class $E(S)$ was obtained by Dvurečenskij and Pulmannova in [2]. In fact this means that the "orthomodular" definition of a measure for a unitary (but incomplete) space is not very rich in content. (However, note that our Definition and a version of the Hamhalter–Ptak definition in the context of the von Neumann algebra $\mathcal{M}$ are equivalent for the class $L_{\mathcal{M}}(S)$).

References

1. Dorofeev, S.V., Sherstnev, A.N.: Frame type functions and their application. Russ. Math. **34**(4), 23–29 (1990)
2. Dvurečenskij, A., Pulmannova, S.: A signed measure completeness criterion. Lett. Math. Phys. **17**, 253–261 (1989)
3. Hamhalter, J., Ptak, P.: A completeness criterion for inner product spaces. Bull. Lond. Math. Soc. **19**, 259–263 (1987)
4. Pedersen, G.K., Takesaki, M.: The Radon-Nikodym theorem for Neumann algebras. Acta Math. **130**(1–2), 53–87 (1973)